**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH2050A Mathematical Analysis I (Fall 2022)**
**Suggested Solution of Homework 3**

(1) For any $k \in \mathbb{N}$, for any $n \geq k$, by definition, $x_n + y_n \leq \sup_{n\geq k} x_n + \sup_{n\geq k} y_n$. Then $\sup_{n\geq k}(x_n + y_n) \leq \sup_{n\geq k} x_n + \sup_{n\geq k} y_n$. Taking the limit as $k \to \infty$, by the algebraic property of limit, $\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$.

(2) Note that $x_n - x_{n-1} = -\frac{3}{4}(x_{n-1} - x_{n-2})$ for any $n > 2$. By induction, one can show $x_{k+1} - x_k = (-\frac{3}{4})^{k-1}(x_2 - x_1)$ for any $k \in \mathbb{N}$. Then $x_n = x_1 + \sum_{k=1}^{n-1}(x_{k+1} - x_k) = x_1 + (x_2 - x_1)\sum_{k=1}^{n-1}(-\frac{1}{4})^{k-1} = x_1 + \frac{4}{7}(x_2 - x_1)(1 - (-\frac{3}{4})^{n-1})$. Hence, $\lim_{n\to\infty} x_n = x_1 + \frac{4}{7}(x_2 - x_1) = \frac{4}{7}x_2 + \frac{3}{7}x_1$.

(3) Note that $|x_n - x_m| = |\sum_{k=m}^{n-1}(x_{k+1} - x_k)| \leq \sum_{k=m}^{n-1} |x_{k+1} - x_k| \leq \sum_{k=m}^{n-1} r^k < \frac{r^m}{1-r}$ for any $n > m$. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\frac{r^N}{1-r} < \epsilon$. Therefore, $|x_n - x_m| < \epsilon$ for any $n > m > N$, i.e., $\{x_n\}$ is Cauchy. Hence, $\{x_n\}$ is convergent.

(4) Since $\{x_n\}$ is Cauchy, there exists $N \in \mathbb{N}$ such that for any $n > m > N$, $|x_n - x_m| < 1$. If $x_n \neq x_m$, then $|x_n - x_m| \geq 1$. Thus $x_n = x_m$ for any $n > m > N$, i.e., $x_n$ is a constant for $n > N$.